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2008.105

Version révisée

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SCIENTIFIQUE

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://ces.univ-paris1.fr/cesdp/CES-docs.htm
ISSN : 1955-611X

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(February 2009)

Abstract

Our earlier papers [2,3,4,5,6] had extended to asymmetric information some classical existence theorems of general equilibrium theory [1,9,10], under the standard assumption that agents had perfect foresights, that is, they knew which price would prevail tomorrow on each spot market. Yet, observation suggests that agents more often trade with an unprecise knowledge of future prices, that is, have no clear price model. To meet this observation, we now consider a pure exchange financial economy where agents' anticipations are represented by idiosyncratic sets of plausible prices, which intersect on each spot market. A state of equilibrium is reached, in this economy, when agents' anticipations include all 'true' spot prices and their trade and consumption decisions are optimal, given budget constraints, and clear on all markets ex post. Extending the result of [4], we show the existence of this 'correct foresights equilibrium' is characterized by the no-arbitrage condition of [2].

Key words: general equilibrium, incomplete markets, asymmetric information, arbitrage, existence of equilibrium.

JEL Classification: D52

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Document de Travail du Centre d'Economie de la Sorbonne - 2008.105

1 Introduction

When asymmetric information prevails on financial markets, the existence and value of equilibrium prices and allocations depend on how agents update their beliefs from observing markets. In a traditional response to that problem, called '*rational expectations*', agents would refer, quoting Radner, 1979, [13], to "*a 'model' or 'expectations' of how equilibrium prices are determined*".

In our approach [2,3], agents having asymmetric information and no price expectations along Radner would update their beliefs from analysing arbitrage on financial markets. A broad concept of equilibrium would embed, as two particular applications, the classical financial equilibrium, and equilibrium with asymmetric information. Assessing the existence properties of that equilibrium, we showed in [4,5,6] that the classical results of symmetric information, e.g., Cass, 1984 [1], for nominal assets, Geanakoplos-Polemarchakis, 1986 [10], for numeraire assets, Duffie-Shafer, 1985 [9], for real assets, would extend to asymmetric information, namely, that a financial equilibrium (or '*pseudo-equilibrium*' in case of real assets) would always exist under a no-arbitrage condition, whatever agents' information.

So far, we had retained the standard assumption that agents believed with certainty which commodity price would prevail tomorrow, conditionnally on the random state. Single pricing, better known as '*perfect foresights*' when all price expectations are correct, is a common feature of current general equilibrium notions, including Arrow-Debreu's, Radner's sequential equilibrium, Grandmont's temporary equilibrium [11], or Radner's rational expectations equilibrium with asymmetric information. Yet, common observation suggests that agents more often trade with no precise knowledge or certainty, ex ante, about future spot prices.

Somehow, this single pricing assumption embeds the idea that agents have a price model in any standard sequential equilibrium, akin to Radner's [13] in the case of asymmetric information. We showed in [2,4] that a sequential equilibrium with asymmetric information was well defined and existed in the absence of rational expectations. We now generalize this result by also dropping perfect foresights. Indeed, an exhaustive general equilibrium model, in which agents could be both asymmetrically informed and truly unaware of how equilibrium prices are determined, should drop both assumptions of rational expectations and perfect foresights.

The model we propose is the simplest setting to deal with that problem, namely, a two-period pure exchange economy, where each agent anticipates, in each future state, a set of plausible prices, called price expectations, endowed with a probability distribution. Such expectations and probabilities are specific to each agent and given at the first period. When all consumers have, at least, one expectation in common on each spot market (a consistency condition of any sequential equilibrium), their price expectations are said to define a '*structure of beliefs*'.

The rest of the model is akin to [2], namely, there is an incomplete market of nominal assets, which permits limited financial transfers across periods and states, and on which agents may be asymmetrically informed, and a commodity market, for the purpose of consumption, which consists in the spot markets of the two periods.

The equilibrium concept of the model embeds and extends that of [2] (in which agents anticipate exactly one price), by replacing the condition of perfect foresights by a milder one of '*correct price foresights*', along which the '*true*' spot prices (those which may prevail tomorrow) are in all agents' expectations sets. A state of equilibrium, or '*correct foresights equilibrium*' (C.F.E.), is reached when agents have correct price foresights and make (ex ante) trade and consumption decisions, which are optimal in the budget sets and clear on all markets at both periods.

We show as our main Theorem, under mild conditions on agents' structure of beliefs, that the existence of that sequential equilibrium (C.F.E.) is still characterized by the no arbitrage condition of [2]. Extending the existence theorem of [4], the same result was proved in [7], in the case of finite price expectations sets.

Though the existence properties are similar, the values of equilibrium prices and allocations may differ from those of the standard model (with perfect foresights). Indeed, equilibrium values in our model depend on agents' structure of beliefs, that is, on how each agent anticipates prices at the first period, and not only on the fundamentals of the economy (wealth, preferences, transfer opportunities, ...), as in the classical general equilibrium model.

Thus, we show that agent's private information, idiosyncratic beliefs, or uncertainty about future prices, would not affect the existence, but the value of equilibrium prices and allocations. This suggests how erroneous expectations today may affect equilibrium tomorrow, and account for phenomena such as speculation, crash, volatility, or bubbles, which are a puzzle to the classical equilibrium theory, where agents perfectly anticipate prices.

The paper is organized as follows. Section 2 presents the basic model, in which agents may have asymmetric information and infinitely many price expectations in each anticipated state. It defines the notions of payoff and information structure, of structure of beliefs and of correct foresights equilibrium. It states the existence Theorem. Section 3 proves the Theorem. Section 4 concludes. An Appendix proves technical Lemmas.

2 The basic model

We consider a pure-exchange financial economy with two periods ($t \in \{0,1\}$) and two markets, a purely financial market and a commodity market. There is an a priori uncertainty at the first period ($t = 0$) about which state s of a given state space S will prevail at the second period ($t = 1$), when all uncertainty is removed. The economy is finite, in the sense that the sets of agents (or consumers), $I := \{1, ..., m\}$, of commodities, $\{1, ..., L\}$, states of nature, S, and financial assets, $\{1, ..., J\}$, are finite.

Hereafter, we present successively the notations used throughout the paper, the markets and agents' beliefs, consumers' behavior, the notion of correct foresights equilibrium and its existence property.

2.1 the model's notations

Throughout the paper, we shall denote by $\cdot$ and $\|.\|$, respectively, the scalar product and Euclidean norm of an Euclidean space and, for every non-empty set E, by $\mathcal{P}(E)$ the set of non-empty subsets of E. We shall denote by $s = 0$ the non-random state of nature at $t = 0$ and let $\Sigma' := \{0\} \cup \Sigma$, for each subset Σ of S.

For all sets $\widetilde{\Sigma} \in \mathcal{P}(S')$ and $\Sigma \in \mathcal{P}(\widetilde{\Sigma})$, for every $S \times J$-matrix $V := (v_j[s])_{(s,j) \in S \times J}$ and $\Sigma \times J$-matrix A, for all sets $X \in \mathcal{P}(\mathbb{R}^{L\widetilde{\Sigma}})$ and $X' \in \mathcal{P}(\mathbb{R}^{\widetilde{\Sigma}})$, for all tuples $(q, s, l) \in \mathbb{R}^J \times \Sigma \times \{1, ..., L\}$ and $(x, x', (y, y'), (z, z')) \in \mathbb{R}^{L\widetilde{\Sigma}} \times \mathbb{R}^{\widetilde{\Sigma}} \times (\mathbb{R}^{\Sigma})^2 \times (\mathbb{R}^{L\Sigma})^2$, we shall denote by:

- $x[\Sigma]$ and $x'[\Sigma]$, respectively, the truncations of x on $\mathbb{R}^{L\Sigma}$ and x' on $\mathbb{R}^{\Sigma}$;
- $X[\Sigma] := \{x \in \mathbb{R}^{L\Sigma} : \exists \widetilde{x} \in X, x = \widetilde{x}[\Sigma]\}$;
- $X'[\Sigma] := \{x' \in \mathbb{R}^{\Sigma} : \exists \widetilde{x} \in X', x = \widetilde{x}[\Sigma]\}$;

- $A[s]$, $y[s]$, $z[s]$, resp., the row, scalar, vector, indexed by $s \in \Sigma$, of A, y, z;
- $z^l[s]$ the l^{th} component of $z[s] \in \mathbb{R}^L$ and we let $z^l := (z^l[s]) \in \mathbb{R}^\Sigma$;
- $y \leqslant y'$ and $z \leqslant z'$ (resp. $y << y'$ & $z << z'$) the relationships $y[s] \leqslant y'[s]$ and $z^l[s] \leqslant z'^l[s]$ (resp. $y[s] < y'[s]$ and $z^l[s] < z'^l[s]$) for every $(l, s) \in \{1, ..., L\} \times \Sigma$;
- $y < y'$ (resp. $z < z'$) the relationships $y \leqslant y'$, $y \neq y'$ (resp. $z \leqslant z'$, $z \neq z'$);
- $z \,\square\, z'$ the vector $(z[s] \cdot z'[s]) \in \mathbb{R}^\Sigma$, $y \,\square\, z$ the vector $(y[s] z[s]) \in \mathbb{R}^{L\Sigma}$;
- $V(\Sigma)$ (when $0 \notin \Sigma$) the $\Sigma \times J-$matrix s.t. $V(\Sigma)[s] := V[s]$, for each $s \in \Sigma$;
- $W(\Sigma, q)$ (when $0 \notin \Sigma$) the $\Sigma' \times J-$matrix such that $W(\Sigma, q)[0] := -q$ and, for each $s \in \Sigma$, $W(\Sigma, q)[s] := V[s]$ and we let $W(q) := W(S, q)$;
- $\mathbb{R}^{L\Sigma}_+ := \{x \in \mathbb{R}^{L\Sigma} : x \geqslant 0\}$ and $\mathbb{R}^\Sigma_+ := \{x \in \mathbb{R}^\Sigma : x \geqslant 0\}$, $\mathbb{R}^{L\Sigma}_{++} := \{x \in \mathbb{R}^{L\Sigma} : x >> 0\}$ and $\mathbb{R}^\Sigma_{++} := \{x \in \mathbb{R}^\Sigma : x >> 0\}$;
- $\Re^{L\tilde{\Sigma}}_+$ the Borel set of $\mathbb{R}^{L\tilde{\Sigma}}_+$.

We now present the markets and consumer's beliefs and behavior.

2.2 Information, anticipations and markets

2.2.1 Information and anticipations

At $t = 0$, each agent $i \in I$ receives or infers a private information signal, or set, $S_i \subset S$, along which the true state will be in S_i. Henceforth, we let $(S_i) = (S_i)_{i=1}^m$ be given, called the information structure, and $\underline{\mathbf{S}} := \cap_{i=1}^m S_i$ be agents' pooled information set, which is assumed to contain the true state. Thus, along all agents' information at $t = 0$, each state $s \in \underline{\mathbf{S}}$ (and only those states) may prevail at $t = 1$. Yet, each agent $i \in I$ sees all states $s \in S_i$ as realizable, i.e., has incomplete information if $S_i \neq \underline{\mathbf{S}}$.

At $t = 0$, only a fictitious observer would normally know, from the economy and agents' beliefs and characteristics, what the price $p[s] \in \mathbb{R}^L_+$ of the L commodities would be if a given state $s \in \underline{\mathbf{S}}$ prevailed at $t = 1$. Yet, this price exists at equilibrium.

In this model, agents need not know, or anticipate with certainty, any future price. Thus, they need not have a unique price anticipation in any future state, as was the case in general equilibrium models so far - whether Grandmont's temporary equilibrium, Radner's rational expectations equilibrium with asymmetric information, or the classical models of symmetric information and perfect price foresights.

Instead, we assume that agents have (at $t = 0$) a set of plausible prices for the second period (endowed with a probability distribution), called '*price expectations*', which overlap across agents on each spot market. Formally:

Definition 1 *For every pair $(i, s) \in I \times S_i$, let $\pi_{(i,s)}$ be a given probability on $(\mathbb{R}^L_+, \Re^L_+)$, let $\pi_i := \odot_{s \in S_i} \pi_{(i,s)}$ be the product probability on $(\mathbb{R}^{LS_i}_+, \Re^{LS_i}_+)$ of the $\pi_{(i,s)}$ (for $s \in S_i$) and consider the following sets, some of which may be empty:*

$B(p, \frac{1}{n}) := \{\overline{p} \in \mathbb{R}^L_+ : \|\overline{p} - p\| < \frac{1}{n}\}$, *for every* $(p, n) \in \mathbb{R}^L_+ \times \mathbb{N}^*$;

$P_{(i,s)} := \{p \in \mathbb{R}^L_+ : \pi_{(i,s)}(B(p, \frac{1}{n})) > 0, \forall n \in \mathbb{N}^*\}$;

$P^*_{(i,s)} := \{p \in P_{(i,s)} : \pi_{(i,s)}(\{p\}) := [\lim_{n \to \infty} \downarrow \pi_{(i,s)}(B(p, \frac{1}{n}))] > 0\}$;

$P_{(i,0)} := \{p \in \mathbb{R}^L_+ : \|p\| \leqslant 1\}$, $P^o_i := \Pi_{s \in S'_i} P_{(i,s)}$, $P_i := \Pi_{s \in S_i} P_{(i,s)}$ *and* $P^*_i := \Pi_{s \in S_i} P^*_{(i,s)}$.

The collection $\pi := (\pi_i)$ is said to be a structure of beliefs if $\cap_{i=1}^m P_i[\underline{\mathbf{S}}]$ is non-empty. Given $\pi := (\pi_i)$, a structure of beliefs, $i \in I$ and $s \in S_i$, we shall refer to P_i as the i^{th} agent's price expectations set and to $P_{(i,s)}$ as her price expectations set in state s.

Henceforth, we assume that agents are endowed with a structure of beliefs, $\pi := (\pi_{(i,s)})$, which is fixed and always referred to, jointly with the sets of Definition 1.

2.2.2 Markets

There are two markets: a commodity market and a purely financial market.

As is standard in a pure exchange economy, agents' consumptions are non-negative bundles of the L goods in each expected state. Given her information set S_i, each agent $i \in I$, is granted a random wealth flow, or endowment, $e_i \in X_i := \mathbb{R}_+^{LS'_i}$, which secures delivery of the commodity bundle $e_i[0] \in \mathbb{R}^L$ in the non-random state $s = 0$, and, in each state $s \in S_i$, of the commodity bundle $e_i[s] \in \mathbb{R}^L$ if state s prevails.

However, agents may only exchange commodities on spot markets of realizables states, namely, of the states $s \in \#\underline{\mathbf{S}}'$. The commodity market consists in the $\#\underline{\mathbf{S}}'$ spot markets for commodities, which agents consume, or exchange, at $t = 0$, on the spot market of the non-random state $s = 0$, and, at $t = 1$, on the spot market of the particular state $s \in \underline{\mathbf{S}}$, which will randomly prevail tomorrow. A commodity price, $p \in \mathbb{R}_+^{L\underline{\mathbf{S}}'}$, embeds the $\#\underline{\mathbf{S}}'$ prices $p[s] \in \mathbb{R}_+^L$ (for $s \in \underline{\mathbf{S}}'$), at which commodities would be traded on the state-s spot market.

The financial market permits limited transfers across periods and states, via J nominal assets $j \in \{1, ..., J\}$, whose contingent payoffs, in each state $s \in S$, are denoted by $v_j[s]$ and yield a $S \times J-$matrix $V = (v_j[s])$, henceforth fixed, such that $J = rankV$. With no loss of generality, agents are not endowed with assets, but may exchange portfolios unrestrictively. A portofolio $z := (z^j) \in \mathbb{R}^J$ specifies the quantity z^j of each asset $j \in \{1, ..., J\}$, positive, if purchased, or negative, if sold. Given an asset price $q \in \mathbb{R}^J$, an agent $i \in I$ may thus purchase a portofolio $z := (z^j) \in \mathbb{R}^J$ for $q \cdot z$ units of account at $t = 0$, against the expected flow $V(S_i)z$ of payoffs at $t = 1$.

We refer to the pair $[V, (S_i)]$ as the payoff and information structure. One particular class of structures $[V, (S_i)]$ is of interest: those for which there exists a price

$q \in \mathbb{R}^J$ and a collection of weights $(\lambda_i) \in \Pi_{i=1}^m \mathbb{R}_{++}^{S_i}$, such that $q = {}^t\lambda_i V(S_i)$, for each $i \in I$. Then, by a standard separation argument, the financial market grants no agent an arbitrage at price q, that is, the possibility of a positive money transfer, in one state, at no cost in any other. When that condition holds, the structure $[V, (S_i)]$ is said to be (q-)arbitrage-free (see [2]).

Market prices may be bounded to one on each spot market at no cost. We will therefore restrict admissible prices to the following convex compact sets:

$\Lambda := \{p \in \mathbb{R}_+^{L\underline{\mathbf{S}}'} : \|p[s]\| \leqslant 1,\ \forall s \in \underline{\mathbf{S}}'\}$ and $\mathcal{M} := \{(p,q) \in \Lambda \times \mathbb{R}^J : \|q\| \leqslant 1\}$.

And we refer to the following subsets of normalized prices:

$\Delta := \{p \in \Lambda : \|p[s]\| = 1, \forall s \in \underline{\mathbf{S}}\}$ and $\mathcal{M}^* := \{(p,q) \in \mathcal{M} : p \in \Delta, \|p[0]\| + \|q\| = 1\}$;

$\Delta_\varepsilon := \{p \in \Delta : p^l[s] \geqslant \varepsilon,\ \forall (s,l) \in \underline{\mathbf{S}} \times \{1,...,L\}\} \neq \varnothing$, for every $\varepsilon \in]0, \frac{1}{L}[$.

2.3 Consumers' behavior

Each agent $i \in I$ makes her consumption plans at $t = 0$ for the first period, and for each anticipated state $s \in S_i$ and price $p_i[s] \in P_{(i,s)}$ of the second period. Thus, her set of consumption plans is:

$Y_i := L^c(P_{(i,0)}, \mathbb{R}_+^L) \times \Pi_{s \in S_i} L^0(P_{(i,s)}, \mathbb{R}_+^L)$,

where $L^c(P_{(i,0)}, \mathbb{R}_+^L)$, identified to $\mathbb{R}_+^L$, stands for the set of constant mappings from $P_{(i,0)} = \Lambda[0]$ to $\mathbb{R}_+^L$, and, for each $s \in S_i$, $L^0(P_{(i,s)}, \mathbb{R}_+^L)$ denotes the set of continuous mappings from $P_{(i,s)}$ to $\mathbb{R}_+^L$. The economic interpretation of Y_i is the following: a consumption plan $y \in Y_i$ embeds a non-random consumption decision $y(p[0]) \in \mathbb{R}_+^L$ at $t = 0$, which does not depend on $p[0] \in \Lambda[0]$ and is also denoted by $y[0]$, and it relates continuously, every expectation $p_i \in P_i$ and state $s \in S_i$, to a conditional consumption decision $y(p_i[s]) \in \mathbb{R}^L_+$ in state s at $t = 1$. That consumption $y(p_i[s])$ is conditional on both conditions that state $s \in S_i$ and price $p_i[s]$ prevailed at $t = 1$. Thus, the plan y relates continuously every price $p_i \in P_i^o$ to a collection $y(p_i) := (y(p_i[s])) \in \mathbb{R}^{LS_i'}_+$ of (conditional) consumption decisions across states $s \in S_i'$ (with $y(p_i[0])$ constant).

We assume that each agent $i \in I$ is endowed with ordered separable preferences across states, namely, with continuous utility indexes $u_i^s : \mathbb{R}^2_+ \to \mathbb{R}_+$, for each $s \in S_i$, which define her utility function, U_i, and preference correspondence, $\mathcal{R}_i$, as follows:

$$U_i : y \in Y_i \mapsto U_i(y) := \sum_{s \in S_i} \int_{p \in P_i} u_i^s(y[0], y(p[s])) d\pi_{(i,s)}(p[s]);$$
$$\mathcal{R}_i : y \in Y_i \mapsto \mathcal{R}_i(y) := \{y' \in Y_i : U_i(y') > U_i(y)\}.$$

The above notion of consumption plan embeds and extends the classical one (and that of [4]), in which the price expectation $p[s]$ at $t = 1$ and consumption decision $y(p[s])$ are unique in each $s \in S_i'$ (hence, Y_i identifies to $\mathbb{R}^{LS_i'}_+$).

The collection $\mathcal{F} := ([V, (S_i)], (e_i), (u_i^s))$ of a structure $[V, (S_i)]$, of endowments $(e_i) := (e_i)_{i \in I}$ and utility indexes, $(u_i^s) := (u_i^s)_{(i,s) \in I \times S_i}$, is henceforth fixed and referred to as the fundamentals, and distinguished from the structure, $\pi := (\pi_{(i,s)})$, of beliefs.

Taking the observed prices as given, agents make their trade and consumption plans at $t = 0$, so as to optimize their welfare, facing budget constraints in all state and price they observe (at $t = 0$) or expect (for $t = 1$). Thus, given the observed market prices $(p[0], q) \in \mathcal{M}[0]$ at $t = 0$, the generic consumer $i \in I$ chooses a strategy $(y, z) \in Y_i \times \mathbb{R}^J$, which maximizes her utility function U_i in the following budget set:

$$B_i(p[0], q) := \{(y, z) \in Y_i \times \mathbb{R}^J : p_i \,\square\, (y(p_i) - e_i) \leqslant W(q, S_i)z, \forall p_i \in \{p[0]\} \times P_i\},$$

that is, a strategy which belongs to the set $B_i^*(p[0], q) := \arg\max_{(y,z) \in B_i(p[0],q)} U_i(y)$.

2.2 Definition and existence of a 'correct foresights equilibrium'

2.2.1 The notion of correct foresights equilibrium

An allocation is a collection $y := (y_i) \in Y := \Pi_{i=1}^m Y_i$ of consumption plans. Given a market price $(p,q) \in \mathcal{M}$, such that $p[\underline{\mathbf{S}}] \in \cap_{i=1}^m P_i[\underline{\mathbf{S}}]$, we define the following sets of attainable allocations, portfolios and strategies:

$\mathcal{A}(p) := \{y := (y_i) \in Y : \sum_{i=1}^m (y_i(p[s]) - e_i[s]) = 0, \ \forall s \in \underline{\mathbf{S}}'\}$;

$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0\}$;

$\mathcal{Y}(p,q) := \{[(y_i, z_i)] \in \Pi_{i=1}^m B_i(p[0], q) : (y_i) \in \mathcal{A}(p), \ (z_i) \in \mathcal{Z}\}$.

And we let $\mathcal{Y}^*(p,q) := \mathcal{Y}(p,q) \cap \Pi_{i=1}^m B_i^*(p[0],q)$ be the set of optimal attainable strategies. The economy defined above for the given fundamentals, $\mathcal{F} := ([V,(S_i)],(e_i),(u_i^s))$, and structure of beliefs, $\pi := (\pi_i)$, is denoted by $\mathcal{E}_{(\mathcal{F},\pi)}$. At a sequential equilibrium of this economy, all agents should have forseen the true spot prices and made trade and consumptions decisions, at $t = 0$, which clear on all markets at both periods and optimise their welfare within the budget sets. Formally:

Definition 2 *A price and strategy collection,* $((p,q),[(y_i,z_i)]) \in \mathcal{M} \times \Pi_{i=1}^m B_i(p[0],q)$, *is an equilibrium of the economy* $\mathcal{E}_{(\mathcal{F},\pi)}$, *or correct foresights equilibrium (CFE), if:*

(a) $p \in \cap_{i=1}^m P_i^o[\underline{\mathbf{S}}']$ *along Definition 1;*

(b) $\forall i \in I, \ B_i(p[0],q) \cap \mathcal{R}_i(y_i) \times \mathbb{R}^J = \varnothing$*;*

(c) $(y_i) \in \mathcal{A}(p)$*;*

(d) $(z_i) \in \mathcal{Z}$*.*

Along Definition 2, at equilibrium, agents never need revise their anticipations or decisions ex post, so as to improve their welfare or let markets clear. Indeed, for every $s \in \underline{\mathbf{S}}$, the true spot price, $p[s]$, belongs to every agent's expectations set from

Condition (a) of Definition 2 and all agents' equilibrium consumption decisions (at $t=0$) are optimal (ex ante and ex post) and let all spot markets clear.

This outcome is the main difference with the notion of temporary equilibrium along Grandmont, where agents' ex ante decisions and expectations need not be optimal, correct or market clearing ex post. However, we will show that both classical notions of sequential and temporary equilibrium are limit cases of that of C.F.E.

2.2.2 The existence Theorem

Extending the result of [4], the following Theorem characterizes the existence of equilibrium by the no-arbitrage condition of [2]. First, we state a Lemma.

Lemma 1 *For every* $(i,\eta,(p,q)) \in I\times\mathbb{R}_{++}\times\mathcal{M}$, *we consider the following sets:*

$B_i(p[0],q,\eta) := \{(y,z)\in Y_i\times\mathbb{R}^J : p_i[s]\cdot(y(p_i[s])\text{-}e_i[s]) \leqslant -\eta+W(q)[s]\cdot z,\ \forall p_i \in \{p[0]\}\times P_i, \forall s \in S_i'\}$;

$\mathcal{A}^-(p) := \{y := (y_i) \in Y : \sum_{i=1}^m (y_i(p[s]) - e_i[s]) \leqslant 0,\ \forall s \in \underline{\mathbf{S}}'\}$;

$\mathcal{Z}^1 := \{z := (z_i) \in \mathcal{Z} : V[s_i]\cdot z_i \geqslant -1,\ \forall(i,s_i) \in I\times S_i\}$.

Assume that, for each $(i,s) \in I\times\underline{\mathbf{S}}$, *the utility index* u_i^s *is* C^1 *and such that* $\frac{\partial u_i^s}{\partial y^l}(y) > 0$, *for all* $(y,l) \in \mathbb{R}_+^{2L}\times\{1,...,2L\}$. *Then, there exists* $(r^1,r^2,\varepsilon_0) \in \mathbb{R}_{++}^3$, *which only depends on the fundamentals* $\mathcal{F} := ([V,(S_i)],(e_i),(u_i^s))$, *such that the following Assertions hold:*

(i) $(p \in \Lambda$ *and* $(y_i) \in \mathcal{A}^-(p)) \Rightarrow (\sum_{(i,s)\in I\times\underline{\mathbf{S}}'} \|y_i(p[s])\| < r^1)$;

(ii) $(z := (z_i) \in \mathcal{Z}^1) \Rightarrow (\|z\| := \sum_{i\in I}\|z_i\|) < r^2)$;

(iii) $((p,q) \in \mathcal{M}^*,\ p \in \cap_{i=1}^m P_i^o[\underline{\mathbf{S}}']$ *and* $\mathcal{Y}^*(p,q) \neq \varnothing) \Rightarrow (p \in \Delta_{\varepsilon_0})$;

(iv) $((i,\varepsilon,(p,q)) \in I\times\mathbb{R}_{++}\times\mathcal{M}^*$ *and* $p_i^l[s] \geqslant \varepsilon,\ \forall(p_i,l,s) \in P_i\times\{1,...,L\}\times S_i)$

$\Rightarrow (\exists\eta > 0 : B_i(p[0],q,\eta) \neq \varnothing)$.

Proof see the Appendix. □

Henceforth, we assume that agents' preferences meet the Conditions of Lemma 1 and set as given bounds $(r^1,r^2,\varepsilon_0) \in \mathbb{R}_{++}^3$, which satisfy the Assertions of Lemma

1. We say the economy $\mathcal{E}_{(\mathcal{F},\pi)}$ is standard if it meets the following Assumptions, which will be discussed in Section 4.

- ***Assumption A1:*** $\forall i \in I,\ e_i >> 0$;
- ***Assumption A2:*** $\exists \varepsilon > 0, \exists M > 1 : \forall (i, p_i, s, l) \in I \times P_i \times S_i \times \{1, ..., L\},\ \varepsilon < p_i^l[s] < M$;
- ***Assumption A3:*** $\forall (i, s) \in I \times S_i,\ u_i^s$ *is* C^1*, strictly concave, strictly increasing, that is,* $\frac{\partial u_i^s}{\partial y^l}(y) > 0,\ \forall (y, l) \in \mathbb{R}_+^{2L} \times \{1, ..., 2L\}$;
- ***Assumption A4:*** $\Delta_{\varepsilon_0} \subset (\cap_{i=1}^m P_i^o[\underline{\mathbf{S}}'])$.

The following Theorem characterizes the existence of equilibrium.

Theorem 1 *A standard economy* $\mathcal{E}_{(\mathcal{F},\pi)}$ *admits an equilibrium if and only if its payoff and information structure,* $[V, (S_i)]$*, is arbitrage-free in the sense of [2].*

Before proving that every standard economy $\mathcal{E}_{(\mathcal{F},\pi)}$ with an arbitrage-free structure $[V, (S_i)]$ admits an equilibrium, Claim 1 provides a converse result.

Claim 1 *Let* $((p, q), [(y_i, z_i)]) \in \mathcal{M} \times \Pi_{i=1}^m B_i(p[0], q)$ *be an equilibrium of a standard economy* $\mathcal{E}_{(\mathcal{F},\pi)}$*. Then,* $p >> 0$ *and* $[V, (S_i)]$ *is (*q*)-arbitrage-free along [2].*

Proof Let an equilibrium $\mathcal{C} := ((p, q), [(y_i, z_i)]) \in \mathcal{M} \times \Pi_{i=1}^m B_i(p[0], q)$ of a standard economy $\mathcal{E}_{(\mathcal{F},\pi)}$ be given.

Assume, by contraposition, that there exists $l \in \{1, ..., L\}$, such that $p^l[0] \leqslant 0$. Then, the consumption plan $y_1^* \in Y_1$, identical to y_1 in all components, but $y_1^{*l}[0] := y_1^l[0] + 1$, satisfies $(y_1^*, z_1) \in B_1(p[0], q)$ and, from Assumption *A3*, $y_1^* \in \mathcal{R}_1(y_1)$. This contradicts the fact that $\mathcal{C}$ meets Condition (b) of Definition 2. This contradiction proves that $p[0] >> 0$ and, from Assumption *A2* and Condition (a) of Definition 2, that $p >> 0$.

Assume, by contraposition, $[V,(S_i)]$ fails to be q-arbitrage-free, i.e., there exists $(i,z,s) \in I \times \mathbb{R}^J \times S'_i$ such that $W(S_i,q)z > 0$ and $W(q)[s]\cdot z > 0$. Let $y \in Y_i$ be defined (from Assumption $A2$ and above) by $y(p_i[\overline{s}]) := y_i(p_i[\overline{s}])$ and $y^l(p_i[s]) := \frac{W(q)[s]\cdot z}{LM} + y^l_i(p_i[s])$, for every $(p_i,l,\overline{s}) \in (\{p[0]\}\times P_i)\times\{1,...,L\}\times S'_i/\{s\}$. Then, $(y, z_i+z) \in B_i(p,q) \cap \mathcal{R}_i(y_i) \times \mathbb{R}^J$, from Assumption $A3$, contradicting the fact that $\mathcal{C}$ meets Condition (b) of Definition 2. $\square$

We henceforth assume that the economy $\mathcal{E}_{(\mathcal{F},\pi)}$ defined above is standard with an arbitrage-free structure, $[V,(S_i)]$, and show that this economy admits an equilibrium.

3 The existence proof

We now define a sequence of auxiliary economies with finite expectations sets, each of which admits an equilibrium along Theorem 1 of [4], and we derive from a sequence of equilibria in each auxiliary economy an equilibrium of the economy $\mathcal{E}_{(\mathcal{F},\pi)}$. First, we introduce sets and notations, which are used throughout.

3.1 Auxiliary sets and notations

For every $(i,s,n) \in I \times S_i \times \mathbb{N}$, we let:

$K^n := \{k^n := (k^n_1,...,k^n_L) \in (\mathbb{N} \cap [0, 2^n - 1])^L\}$;

$I^n_{(i,s,k^n)} := P_{(i,s)} \cap \Pi^L_{l=1}]\frac{k^n_l M}{2^n}, \frac{(k^n_l+1)M}{2^n}]$, possibly empty, for each $k^n := (k^n_1,...,k^n_L) \in K^n$;

$K^n_{(i,s)} := \{k^n \in K^n : I^n_{(i,s,k^n)} \neq \varnothing\}$;

$I^n_{(i,s)} := \{I^n_{(i,s,k^n)} : k^n \in K^n_{(i,s)}\}$.

Then, by induction on $m \in \{0,...,n\}$, we choose iteratively $g^n_{(i,s,k^n)} \in I^n_{(i,s,k^n)}$, for all tuple $(i,s,n,k^n) \in I \times S_i \times \mathbb{N} \times K^n_{(i,s)}$ and define a set $G^n_{(i,s)} := \{g^n_{(i,s,k^n)} \in I^n_{(i,s,k^n)} : k^n \in K^n_{(i,s)}\}$:

- for $m = 0$, we set as given $g^0_{(i,s,0)} \in P_{(i,s)} \neq \varnothing$ and let $G^0_{(i,s)} := \{g^0_{(i,s,0)}\}$.

- given $m < n$ and $G^m_{(i,s)} := \{g^m_{(i,s,k^m)} \in I^m_{(i,s,k^m)} : k^m \in K^m_{(i,s)}\}$, we define the set $G^{m+1}_{(i,s)} := \{g^{m+1}_{(i,s,k^{m+1})} \in I^{m+1}_{(i,s,k^{m+1})} : k^{m+1} \in K^{m+1}_{(i,s)}\}$ as follows:

$$g^{m+1}_{(i,s,k^{m+1})} \begin{cases} \textit{is set equal to } g^m_{(i,s,k^m)} \textit{ if there exists } (k^m, g^m_{(i,s,k^m)}) \in K^m_{(i,s)} \times G^m_{(i,s)} \textit{ s.t. } g^m_{(i,s,k^m)} \in I^{m+1}_{(i,s,k^{m+1})} \\ \textit{is set arbitrary in } I^{m+1}_{(i,s,k^{m+1})} \qquad \textit{if } I^{m+1}_{(i,s,k^{m+1})} \neq \varnothing \textit{ and } I^{m+1}_{(i,s,k^{m+1})} \cap G^m_{(i,s)} = \varnothing \end{cases}$$

Iterating the process, for $m = 0, ..., n-1$, yields $G^n_{(i,s)}$. Then, we let $G^n_i := \Pi_{s \in S_i} G^n_{(i,s)}$, for every $(i,n) \in I \times \mathbb{N}$.

For convenience, we assume costlessly that, for every $(i,s,n,k^n) \in I \times S_i \times \mathbb{N}^* \times K^n_{(i,s)}$, the element $g^n_{(i,s,k^n)} \in G^n_{(i,s)}$ is in the interior of $I^n_{(i,s,k^n)}$.[2] Under that condition, for every $n \in \mathbb{N}^*$, the relation $G^{n-1}_i \subset G^n_i$ holds for every $i \in I$ and there exists $r^n \in]0, \frac{1}{4^n}[$ such that the following conditions hold, for every $(i,s,k^n,p) \in I \times S_i \times K^n_{(i,s)} \times \Lambda$:

$I^n_{(i,s,k^n)} / B^{r^n}_i(p[s]) \neq \varnothing$, where $B^{r^n}_i(p[s]) := \{\overline{p} \in P_{(i,s)} / P^*_{(i,s)} : \|\overline{p} - p[s]\| < r^n\}$.

We henceforth set as given a decreasing sequence $(r^n) \in \mathbb{R}^{\mathbb{N}^*}_{++}$ of elements $r^n \in]0, \frac{1}{4^n}[$ (for $n \in \mathbb{N}^*$) which meet the above conditions. For every $(i,s,n,p) \in I \times S_i \times \mathbb{N}^* \times \Lambda$, we let $B^{r^n}_i(p[s]) := \{\overline{p} \in P_{(i,s)} / P^*_{(i,s)} : \|\overline{p} - p[s]\| < r^n\}$, if $s \in \underline{\mathbf{S}}$, and define a discrete probability $\pi^n_{(i,p,s)}$ on $\mathbb{R}^L_+$ as follows:

- $\pi^n_{(i,p,s)}(p[s]) := \pi_{(i,s)}(B^{r^n}_i(p[s]))$, if $s \in \underline{\mathbf{S}}$;
- $\pi^n_{(i,p,s)}(g^n_{(i,s,k^n)}) := \pi_{(i,s)}(I^n_{(i,s,k^n)} / B^{r^n}_i(p[s]))$, for each $k^n \in K^n_{(i,s)}$, if $s \in \underline{\mathbf{S}}$;[3]
- $\pi^n_{(i,p,s)}(g^n_{(i,s,k^n)}) := \pi_{(i,s)}(I^n_{(i,s,k^n)})$, for each $k^n \in K^n_{(i,s)}$, if $s \in S_i / \underline{\mathbf{S}}$.

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[2] If this is not the case, we may always shift above the upper boundary of $I^n_{(i,s,k^n)}$ by an infinitesimal amount to obtain interior points, without changing any result.

[3] If $p[s] \in G^n_{(i,s)}$, i.e., $g^n_{(i,s,k^n)} = p[s]$, for some $k^n \in K^n_{(i,s)}$, we consider costlessly that $p[s]$ and $g^n_{(i,s,k^n)}$ are two different (differenly indexed) elements.

3.2 Auxiliary economies $\mathcal{E}[V^n, (S_i^n)]$

The basic model and the probabilities $\pi^n_{(i,p,s)}$ (for $(i,p,s,n) \in I \times \Lambda \times S_i \times \mathbb{N}^*$) yield, for each $n \in \mathbb{N}^*$, an auxiliary economy, referred to as the n-economy, and described hereafter, with reference to the following notations (for each $(i,j) \in I \times \{1,...,J\}$):

$\widetilde{S}_i^n := \{s^n := (s, p_i[s]) : s \in S_i,\ p_i \in G_i^n\}$ and $S_i^n := \underline{\mathbf{S}} \cup \widetilde{S}_i^n$ and $S^n := \cup_{i=1}^m S_i^n$;

$S_i^n(s) := \{s^n := (s, p_i[s]) \in \widetilde{S}_i^n\} \cup (\{s\} \cap \underline{\mathbf{S}})$, for each $s \in S_i$;

$e_i^n[0] := e_i[0]$, $e_i^n[s^n] := e_i[s]$ and $v_j^n[s^n] := v_j[s]$, for each pair $(s, s^n) \in S_i \times S_i^n(s)$.

This n-economy has two periods ($t \in \{0,1\}$), with an a priori uncertainty at the first period ($t = 0$) about which state $s^n \in S^n$ will prevail at the second period ($t = 1$). As in Section 2, m agents, or consumers, $i \in I := \{1,...,m\}$ may exchange the L goods, $l \in \{1,...,L\}$, on the spots markets of the two periods and J nominal assets, $j \in \{1,...,J\}$, whose flows of payoffs $v_j^n[s^n]$ in each state $s^n \in S^n$ at $t = 1$ are summarized by the $S^n \times J$ payoff matrix $V^n := (v_j^n[s^n])$, defined from above.

Each consumer $i \in I$, receives the private information signal $S_i^n \subset S^n$ at $t = 0$, and has the endowment $e_i^n \in \mathbb{R}_{++}^{LS_i^{n\prime}}$, granting the bundle $e_i^n[s^n] \in \mathbb{R}_{++}^L$ of the L goods in each state $s^n \in S_i^{n\prime} := \underline{\mathbf{S}}' \cup \widetilde{S}_i^n$. We consider costlessly (see footnote 3), that $\widetilde{S}_i^n$ is, formally, an idiosyncratic set of states of the i^{th} agent, none of which will prevail at $t = 1$. Thus, in the n-economy, given $(p,q) \in \mathcal{M}$, agents' pooled information set is $\underline{\mathbf{S}}$, identified to the set $\{(s, p[s]) : s \in \underline{\mathbf{S}}\}$ of true states $s \in \underline{\mathbf{S}}$ and joint true spot price $p[s]$.

For each $(i, (p,q)) \in I \times \mathcal{M}$, using similar notations as in Section 2, we let the i^{th} agent's consumption set, budget set, utility function, preference set be, respectively:

$X_i^n := \mathbb{R}_+^{LS_i^{n\prime}} := \{(x[s^n])_{s^n \in S_i^{n\prime}} : x[s^n] \in \mathbb{R}_+^L, \forall s^n \in S_i^{n\prime} := \underline{\mathbf{S}}' \cup \widetilde{S}_i^n\}$;

$B_i^n(p,q) := \{(x,z) \in X_i^n \times \mathbb{R}^J$ *such that* $p[s]\cdot(x[s]$-$e_i^n[s]) \leqslant W^n(q)[s]\cdot z := W(q)[s]\cdot z$,

for every $s \in \underline{\mathbf{S}}'$, *and* $p_i[s]\cdot(x[s^n]$-$e_i^n[s^n]) \leqslant V^n[s^n]\cdot z = V[s]\cdot z$, *for every* $s^n = (s, p_i[s]) \in \widetilde{S}_i^n\}$;

$U_i^n(p,\cdot) : x \in X_i^n \mapsto U_i^n(p,x) := \sum_{s^n=(s,p_i[s])\in \widetilde{S}_i^n} \pi^n_{(i,p,s)}(p_i[s]).u_i^s(x[0],x[s^n]) + \sum_{s\in\underline{\mathbf{S}}} \pi^n_{(i,p,s)}(p[s]).u_i^s(x[0],x[s])$;

$\mathcal{R}_i^n(p,x) := \{x' \in Y_i^n : U_i^n(p,x') > U_i^n(p,x)\}$, for every $x \in X_i^n$.

And we let: $\mathcal{A}^n := \{x := (x_i) \in X^n := \Pi_{i=1}^m X_i^n : \sum_{i=1}^m (x_i[s]\text{-}e_i[s]) = 0,\ \forall s \in \underline{\mathbf{S}}'\}$,

$\mathcal{Z} := \{(z_i) \in (\mathbb{R}^J)^m : \sum_{i=1}^m z_i = 0\}$,

$\mathcal{Y}^n(p,q) := \{[(x_i,z_i)] \in \Pi_{i=1}^m B_i^n(p,q) : (y_i) \in \mathcal{A}^n,\ (z_i) \in \mathcal{Z}\}$,

be, respectively, the sets of attainable allocations, portfolios and strategies. By construction, and from Assumption *A3*, the mapping $U_i^n : (p,x) \in \Lambda \times X_i^n \mapsto U_i^n(p,x)$ is continuous, hence, the correspondence $\mathcal{R}_i^n : (p,x) \in \Lambda \times X_i^n \mapsto \mathcal{R}_i^n(p,x)$ is continuous.

As is tedious but immediate, we let the reader check, from the above definitions, Assumptions *A1* to *A4*, and the fact that $[V,(S_i)]$, is arbitrage-free, that the n-economy is a standard economy $\mathcal{E}[V^n,(S_i^n)]$ along the definition and notations of [4], and has an arbitrage-free structure $[V^n,(S_i^n)]$, with the only difference that the preference correspondences, $\mathcal{R}_i^n$ (for each $i \in I$), depend continuously on the market price $p \in \Lambda$ in the n-economy (whereas they were price independent in [4]). Similarly, we let the reader check that this sole difference does not affect the validity of any Claim in [4], that is, all Claims of [4] hold, mutatis mutandis, in the n-economy.

Consequently, from Theorem 1, p. 261, of [4], the n-economy admits an equilibrium (henceforth called n-equilibrium), namely, a collection of prices and strategies:

$\mathcal{C}^n := ((p^n,q^n),[(x_i^n,z_i^n)]) \in \mathcal{M}^* \times \mathcal{Y}^n(p^n,q^n)$,

such that $B_i^n(p^n,q^n) \cap \mathcal{R}_i^n(p^n,x_i^n) \times \mathbb{R}^J = \varnothing$, for each $i \in I$.

We now set as given one such equilibrium $\mathcal{C}^n$ for each $n \in \mathbb{N}^*$ and derive from the sequence $(\mathcal{C}^n)_{n\in\mathbb{N}^*}$ an equilibrium of the economy $\mathcal{E}_{(\mathcal{F},\pi)}$.

3.3 An equilibrium of the economy $\mathcal{E}_{(\mathcal{F},\pi)}$

Throughout, for each $n \in \mathbb{N}^*$, we refer to $\mathcal{C}^n := ((p^n, q^n), (x^n, z^n) := [(x_i^n, z_i^n)]) \in \mathcal{M}^* \times \mathcal{Y}^n(p^n, q^n)$ as the n-equilibrium of sub-Section 3.2. The sequence of prices $((p^n, q^n))_{n \in \mathbb{N}^*}$ may be assumed to converge in the compact set $\mathcal{M}^*$. We let $(p^*, q^*) = \lim_{n\to\infty}(p^n, q^n) \in \mathcal{M}^*$. For each $(i, s) \in I \times \underline{\mathbf{S}}'$, the sequence of consumptions $(x_i^n[s]) \in (\mathbb{R}_+^L,)^{\mathbb{N}^*}$ is bounded from Lemma 1-(i), so, may be assumed to converge and we let $x_i^*(p^*[s]) := x_i^*[s] := \lim_{n\to\infty} x_i^n[s]$. Then, the relations $x^n := (x_i^n) \in \mathcal{A}^n$, which hold for each $n \in \mathbb{N}^*$ since $\mathcal{C}^n$ is an n-equilibrium, yield, in the limit: $\sum_{i=1}^m (x_i^*(p^*[s])\text{-}e_i[s]) = 0$, for each $s \in \underline{\mathbf{S}}'$.

The following Lemma provides asymptotic properties of the sequence $(\mathcal{C}^n)$.

Lemma 2 *For every tuple* $(i, p, s, N) \in I \times \Lambda \times S_i \times \mathbb{N}^*$, *we recall that* $G_i^n = \Pi_{s \in S_i} G_{(i,s)}^n$ *and* $P_i^o = \Lambda[0] \times P_i$, *along Definition 1, and we consider the following sets:*

$G_{(i,s)}^\infty := \cup_{n \in \mathbb{N}^*} G_{(i,s)}^n = \lim_{n\to\infty} \uparrow G_{(i,s)}^n$ *and* $G_i^\infty := \cup_{n \in \mathbb{N}^*} G_i^n = \lim_{n\to\infty} \uparrow G_i^n$;

$\mathsf{Arg}_i(p_i) := \{(p_i^n) \in (\mathbb{R}_+^{LS_i'})^{\mathbb{N}^*} : p_i = \lim_{n\to\infty} p_i^n,\ p_i^n \in \Lambda[0] \times G_i^n, \forall n \in \mathbb{N}^*\}$, *for every* $p_i \in P_i^o$.

Then, the following Assertions hold:

(i) $\forall i \in I$, $G_i^\infty \subset P_i$ *and* $\overline{G_i^\infty} = P_i$ *along the above Definition 1*;

(i') $\forall (i, n) \in I \times \mathbb{N}^*$, $(p^n, p^*) \in \Delta_{\varepsilon_0}^2 \subset P_i^o[\underline{\mathbf{S}}']^2$;

(ii) *the sequence* $(z^n) \in \mathcal{Z}^{\mathbb{N}}$ *is bounded, so may be assumed to converge to* $z^* := (z_i^*) \in \mathcal{Z}$;

(ii') $\exists \beta \in \mathbb{R}_{++} : \forall (i, n, s^n, l) \in I \times \mathbb{N}^* \times S_i^n \times \{1, ..., L\}$, $x_i^{nl}[s^n] \leqslant \beta$;

(iii) $\forall (i, p_i, [(p_i^n), (p_i'^n)]) \in I \times P_i^o \times [\mathsf{Arg}_i(p_i)]^2, \exists y_i^*(p_i) \in \mathbb{R}_+^{LS_i'} : y_i^*(p^*[0]) := y_i^*[0] := y_i^*(p_i)[0] = x_i^*[0]$ *and* $y_i^*(p_i[s]) := y_i^*(p_i)[s] = \lim_{n\to\infty} x_i^n[(s, p_i^n[s])] = \lim_{n\to\infty} x_i^n[(s, p_i'^n[s])]$, $\forall s \in S_i$. *Moreover, the mapping* $y_i^* : p_i \in P_i^o \mapsto y_i^*(p_i)$, *as defined above, is continuous, i.e.,* $y_i^* \in Y_i$;

(iii') $\forall (i, s) \in I \times \underline{\mathbf{S}}'$, $y_i^*(p^*[s]) = x_i^*[s] := \lim_{n\to\infty} x_i^n[s]$, *as defined from* (iii) *and above*;

(iv) $\forall i \in I$, $U_i(y_i^*) = \lim_{n\to\infty} U_i^n(p^n, x_i^n) \in \mathbb{R}$, *where* $y_i^* \in Y_i$ *is defined along* (iii).

Proof see the Appendix. □

We can now prove Theorem 1, via the following Claim.

Claim 2 *The prices,* $(p^*, q^*) = \lim_{n\to\infty}(p^n, q^n) \in \mathcal{M}^*$, *portfolios,* $z^* := (z_i^*) := \lim_{n\to\infty} z^n$, *and allocation,* $y^* := (y_i^*) \in Y := \Pi_{i=1}^m Y_i$, *along Lemma 2 define an equilibrium of the economy* $\mathcal{E}_{(\mathcal{F},\pi)}$, *that is,* $\mathcal{C}^* := ((p^*, q^*), [(y_i^*, z_i^*)]) \in \mathcal{M}^* \times \mathcal{Y}^*(p^*, q^*)$ *is a C.F.E.*

Proof Let $\mathcal{C}^* := ((p^*, q^*), [(y_i^*, z_i^*)])$ be defined as in Claim 2. We recall from above that $(p^*, q^*) \in \mathcal{M}^*$ and $z^* \in \mathcal{Z}$. From Lemma 2-(i'), $p^* \in (\cap_{i=1}^m P_i^o[\underline{\mathbf{S}}'])$. From Lemma 2-$(iii')$ and from above, the relation $\sum_{i=1}^m (y_i^*(p^*[s])\text{-}e_i[s]) = 0$ holds for each $s \in \underline{\mathbf{S}}'$. Hence, $\mathcal{C}^*$ meets Conditions (a)-(c)-(d) of the above Definition 2 of equilibrium.

To prove that $\mathcal{C}^*$ is a correct foresights equilibrium it therefore suffices to show that $(y_i^*, z_i^*) \in B_i(p^*[0], q^*)$, for each $i \in I$, and $\mathcal{C}^*$ meets Condition (b) of Definition 2.

Let $i \in I$ and $p_i \in P_i^o := \Lambda[0] \times P_i$ be given. From Lemma 2-(i), $\mathsf{Arg}_i(p_i) \neq \varnothing$. Let $(p_i^n) \in \mathsf{Arg}_i(p_i)$ be given and $y_i^*(p_i) \in \mathbb{R}_+^{LS_i'}$ be defined from the sequences (x_i^n) and (p_i^n) along Lemma 2-(iii). For each $n \in \mathbb{N}^*$, the fact that $\mathcal{C}^n$ is an n-equilibrium implies $p^n[s]\cdot(x_i^n[s] - e_i[s]) \leqslant W(q^n)[s]\cdot z_i^n$, for each $s \in \underline{\mathbf{S}}'$, and $p_i^n[s]\cdot(x_i^n[(s, p_i^n[s])] - e_i[s]) \leqslant V[s]\cdot z_i^n$, for each $s \in S_i$. Hence, passing to the limit, the relations $p^*[s]\cdot(y_i^*(p^*[s]) - e_i[s]) \leqslant W(q^*)[s]\cdot z_i^*$, for each $s \in \underline{\mathbf{S}}'$, and $p_i[s]\cdot(y_i^*(p_i[s]) - e_i[s]) \leqslant W(q^*)[s]\cdot z_i^*$, for each $s \in S_i$, hold, from Lemma 2-(iii) and the continuity of the scalar product. From Lemma 2-(iii)-(iii') and the fact that $p_i \in P_i^o$ was set arbitrary, the latter relations imply: $(y_i^*, z_i^*) \in B_i(p^*[0], q^*)$.

Assume, by contraposition, that $\mathcal{C}^*$ fails to meet Condition (b) of Definition 2, that is, for some $i \in I$, say $i = 1$, there exists $(y_1, z_1) \in B_1(p^*[0], q^*) \cap \mathcal{R}_1(y_1^*) \times \mathbb{R}^J$. Fom Lemma 1-$(iv)$, Assumptions *A2*-*A4* and Lemma 2-(i'), there exist $\eta^0 > 0$ and $(y_1^0, z_1^0) \in B_1(p^*[0], q^*, \eta^0)$. Then, from Assumption *A3* and the convexity of $B_1(p^*[0], q^*)$, for $K \in \mathbb{N}^*$ big enough, the strategy $(y_1^K, z_1^K) := \frac{K-1}{K}(y_1, z_1) + \frac{1}{K}(y_1^0, z_1^0)$ satisfies $(y_1^K, z_1^K) \in$

$B_1(p^*[0], q^*, \frac{\eta^0}{K}) \cap \mathcal{R}_1(y_1^*) \times \mathbb{R}^J$. So, we assume costlessly that $(y_1, z_1) \in B_1(p^*[0], q^*, \eta) \cap \mathcal{R}_1(y_1^*) \times \mathbb{R}^J$, for some $\eta > 0$, and let $\delta > 0$ be such that:

$$U_1(y_1^*) + \delta < U_1(y_1).$$

Using the notations of sub-Section 2.3, we define, for each $n \in \mathbb{N}^*$, the following vector $\overline{y}_1^n \in X_1^n$ and stair function $\widetilde{y}_1^n \in (\mathbb{R}_+^{LS_1'})^{P_1^o}$ by:

- $\overline{y}_1^n[(s, p[s])] := y_1(p[s])$ for each $(s, p) \in S_1 \times G_1^n$ (since, from Lemma 2-(i), $G_1^n \subset P_1$), $\overline{y}_1^n[s] := y_1(p^n[s])$ for each $s \in \underline{\mathbf{S}}'$ (since, from Lemma 2-(i'), $p^n \in P_1^o[\underline{\mathbf{S}}']$);
- for every $p \in P_1^o$, we let $\widetilde{y}_1^n(p[0]) := \widetilde{y}_1^n(p)[0] := y_1[0]$, $\widetilde{y}_1^n(p[s]) := \widetilde{y}_1^n(p)[s] := y_1(p^n[s])$ whenever $(s, p[s]) \in \underline{\mathbf{S}} \times B_1^{r^n}(p^n[s])$, $\widetilde{y}_1^n(p[s]) := \widetilde{y}_1^n(p)[s] := y_1(g^n_{(1,s,k^n)})$ whenever $(s, k^n, p[s]) \in \underline{\mathbf{S}} \times K^n_{(1,s)} \times (I^n_{(1,s,k^n)} / B_1^{r^n}(p^n[s]))$, $\widetilde{y}_1^n(p[s]) := \widetilde{y}_1^n(p)[s] := y_1(g^n_{(1,s,k^n)})$ whenever $(s, k^n, p[s]) \in (S_1 / \underline{\mathbf{S}}) \times K^n_{(1,s)} \times I^n_{(1,s,k^n)}$.

By construction,

$$U_1(\widetilde{y}_1^n) := \sum_{s \in S_1} \int_{p \in P_1} u_1^s(\widetilde{y}_1^n[0], \widetilde{y}_1^n(p[s])) d\pi_{(1,s)}(p[s])$$
$$:= \sum_{s \in (S_1/\underline{\mathbf{S}})} \sum_{k^n \in K^n_{(1,s)}} \pi_{(1,s)}(I^n_{(1,s,k^n)}) u_1^s(y_1[0], y_1(g^n_{(1,s,k^n)}))$$
$$+ \sum_{s \in \underline{\mathbf{S}}} \sum_{k^n \in K^n_{(1,s)}} \pi_{(1,s)}(I^n_{(1,s,k^n)} / B_1^{r^n}(p^n[s])) u_1^s(y_1[0], y_1(g^n_{(1,s,k^n)}))$$
$$+ \sum_{s \in \underline{\mathbf{S}}} \pi_{(1,s)}(B_1^{r^n}(p^n[s])) u_1^s(y_1[0], y_1(p^n[s])) = U_1^n(p^n, \overline{y}_1^n) \text{ for every } n \in \mathbb{N}^*.$$

Since y_1 is uniformly continuous (as continuous on the compact set P_1^o) and $(p^*, q^*) = \lim_{n \to \infty}(p^n, q^n)$, the above relation $(y_1, z_1) \in B_1(p^*[0], q^*, \eta)$ implies:

$$\exists N_1 \in \mathbb{N}^* : (n \geqslant N_1) \Rightarrow ((\overline{y}_1^n, z_1) \in B_1^n(p^n, q^n)).$$

Let $s \in S_1$ be given and consider the following mappings:

$$\varphi_s : p \in P_{(1,s)} \mapsto u_1^s(y_1[0], y_1(p));$$

$\varphi_s^n : p \in P_{(1,s)} \mapsto u_1^s(y_1[0], \widetilde{y}_1^n(p))$, for every $n \in \mathbb{N}^*$.

From the continuity of φ_s on the compact set $P_{(1,s)}$ (and by construction), the relations $\varphi_s^n(p) \leqslant \alpha_s := \sup_{p\in P_{(1,s)}} \varphi_s(p) \in \mathbb{R}$ hold, for every pair $(p,n) \in P_{(1,s)} \times \mathbb{N}^*$. From the uniform continuity of y_1 and u_1^s on a compact set, the relation $\varphi_s(p) = \lim_{n\to\infty} \varphi_s^n(p)$ holds by construction, for every $p \in P_{(1,s)}$. Hence, the sequence of mappings $(\varphi_s^n)_{n\in\mathbb{N}^*}$ and the mapping φ_s satisfy all conditions to apply Lebesgue's theorem of dominated convergence (see, e.g., [12], p. 59), which implies:

$\int_{p\in P_{(1,s)}} \varphi_s(p) d\pi_{(1,s)}(p) = \lim_{n\to\infty} \int_{p\in P_{(1,s)}} \varphi_s^n(p) d\pi_{(1,s)}(p)$, that is,

$\int_{p\in P_1} u_1^s(y_1[0], y_1(p[s])) d\pi_{(1,s)}(p[s]) = \lim_{n\to\infty} \int_{p\in P_1} u_1^s(\widetilde{y}_1^n[0], \widetilde{y}_1^n(p[s])) d\pi_{(1,s)}(p[s])$.

Summing up the above relations for each $s \in S_1$ yields, from the definitions:

$U_1(y_1) = \lim_{n\to\infty} U_1(\widetilde{y}_1^n)$, hence, from above,

$U_1(y_1^*) + \delta < U_1(y_1) = \lim_{n\to\infty} U_1(\widetilde{y}_1^n) = \lim_{n\to\infty} U_1^n(p^n, \overline{y}_1^n)$.

For every $n \geqslant N_1$, the above relation $(\overline{y}_1^n, z_1) \in B_1^n(p^n, q^n)$ and the fact that $\mathcal{C}^n$ is optimal in the n-economy imply $U_1^n(p^n, \overline{y}_1^n) \leqslant U_1^n(p^n, x_1^n)$, hence, from above:

$\exists N_2 \geqslant N_1 : \forall n > N_2,\ U_1(y_1^*) + \frac{\delta}{2} < U_1^n(p^n, x_1^n)$.

The latter relations contradict Lemma 2-(iv). This contradiction completes the proof of Claim 2. Finally, Theorem 1 holds from Claim 1 and Claim 2. □

4 Concluding Remarks

Presenting applications of this model is beyond our purpose and left to a companion paper. Hereafter, we only hint at some applications, arguing that the C.F.E.

concept is related to the standard equilibrium notions and how it may explain phenomena such as speculation, volatility, crashes or bubbles on financial markets.

4.1 Links with the classical sequential and temporary equilibrium

We argue the above model embeds as limit cases and extends the basic notions and properties of both the sequential and the temporary equilibrium.

Our model with finitely many price expectations [7] shows that the classical financial equilibrium is a particular case of correct foresights equilibrium (C.F.E.), in which each agent has a single price expectation, and that the existence of that C.F.E. is characterized by the no-arbitrage condition of [2], as in the classical theory.

Though not our focus, the current paper extends straightforwardly to a notion of temporary equilibrium with price uncertainty. A temporary equilibrium would be defined, here, as a collection of market prices $(p, q) \in \mathcal{M}$, of price expectations along Definition 1 and of optimal strategies, $(y_i, z_i) \in B_i^*(p[0], q)$, for each $i \in I$, which clear on markets at $t = 0$ (i.e., $\sum_{i=1}^m (y_i[0] - e_i[0]) = 0$ and $\sum_{i=1}^m z_i = 0$). From the same arguments as in the proof of Lemma 2-(ii)-(ii'), equilibrium strategies belong to a compact set, under Assumptions *A2-A3* (see Appendix). Given this, all arguments of [4], [7] and above apply (with simplifications) to show that existence of a temporary equilibrium of a standard economy is characterized by the no-arbitrage condition of [2]. This result is unchanged whether agents' expectations sets are finite or not, define a structure of beliefs or not. Again, the above notion and existence claim extend the classical ones of the temporary equilibrium literature [11].

What this paper focussed on was consistent expectations by agents, leading to a sequential equilibrium, with no need for any agent to have a glimpse or a model of how future prices are determined. Here, the structure of beliefs (π_i) is fixed at $t = 0$ and the support (P_i) is independent of (unobserved) prices. The existence of a C.F.E. is guaranteed (under no-arbitrage) by Lemma 1-(iii) and Assumption *A4*.

As shown here, an equilibrium price (whether in the classical model or in ours) always belongs to a compact set (here, Δ_{ε_0}), which only depends on the fundamentals, $\mathcal{F}$, of the economy, and not on agents' beliefs, π. Though not required by the model, we do not rule out that compact set be inferred by an able observer, or even be known by agents trading on markets. Whatsoever, agents have no model of the true price and are uncertain or cautious enough, so as to embed the whole compact set, to which the equilibrium price will belong, into their expectations. Then, along Theorem 1, existence of equilibrium is characterized by the no-arbitrage condition.

This economy seems more realistic than one where agents need have a model (or glimpse) of the true price. Indeed, equilibrium prices depend (continuously) on every agent's beliefs (expectations), which are private. Rational agents, unaware of other agents' beliefs, should keep a continuous set of expectations, which embeds all possible equilibrium prices, as conveyed by Assumption *A4*. Then, a C.F.E. obtains. Of course, expectations sets could be better focussed on true prices. But, this would require agents to have insights, expectations depending on unobserved prices (see [7], Remark 1) or an awareness of other agents' beliefs. Such unrealistic conditions, which are required by [7] and by the classical theory, are dropped here.

4.2 Volatility, bubbles and crashes on markets

Whereas they remain a puzzle to the classical financial equilibrium, which only depends on the fundamentals of the economy, the current model suggests a simple explanation to market phenomena such as speculation, volatility, bubbles and crashes. To show why, we set as given the fundamentals $(V, (e_i), (u_i^s))$ and examine what may happen when agents' structure of information and beliefs, $[(S_i), (\pi_i)]$, varies. Evidence of such phenomena will be provided on simple numerical examples in the companion paper presenting applications of our model.

The C.F.E. depends on the structure $[(S_i), (\pi_i)]$ via budget sets and utility functions. Then, there is no wonder if a commonly shared belief (e.g., in a price increase) becomes self-fulfilling tomorrow. Yet, the model embeds a less intuitive result: equilibrium prices and allocations are unstable relative to the structure of beliefs, π. This is because a continuous shift in π may change the supports, (P_i), discontinuously. For instance, an infinitesimal change in the distribution π_1, may increases the first agent's price expectations set, P_1, by a non-zero measure set. In that case, her budget constraints change and equilibrium may result in completely different values of portfolios, hence, of prices and allocations. We will show on numerical examples that market volatility in times of uncertainty (which affects budget constraints discontinuously), or breaking bubbles, can stem from this mechanism.

A different situation takes place when agents realize they have had mistaken information or beliefs regarding the future. Hereafter, we review three cases.

First, if some agents refine their information up to a new structure $(\Sigma_i) \leqslant (S_i)$, which is not arbitrage-free, all agents are led, in our model, to further refine their information from observing markets (see [3]), until they inferred a new arbitrage-free structure $(S'_i) \leqslant (\Sigma_i)$, which admits another equilibrium. Before the refinement (S'_i) is reached, markets cannot be at equilibrium, and grant better informed agents an arbitrage. However, a crash in asset prices takes place, from which all agents eventually learn information (see [3]). This learning process occurs at the first period $(t = 0)$. Our model could thus account for crash phenomena taking place when new information is inferred by a limited number of traders on financial markets where agents are asymmetrically informed. Then, better informed agents can earn money during a short transitory period leading to a new information structure.

In the two other cases, disequilibrium is met at the second period ($t = 1$).

If agents pooled information set $\underline{\mathbf{S}} := \cap_{i=1}^{m} S_i$, though non-empty, does not contain the true state ex post (i.e., some agents have mistaken information at $t = 0$), markets are at equilibrium at $t = 0$, but not at $t = 1$, in general, and mistaken agents may then face bankruptcy. This mechanism may account for markets (as that of subprimes) upon which risks are underestimated during some period of trade ($t = 0$) by a number of agents. Given market clearance at $t = 0$, there is no *overall* bankruptcy, i.e., global financial transfers break even at $t = 1$. Thus, equilibrium could theoretically be restored via a state policy and public transfers at $t = 1$. We let the reader check that similar remarks apply, mutatis mutandis, to the situation of temporary equilibrium.

Appendix

Herafter, we recall and prove the technical Lemmas of Sections 2 and 3.

Lemma 1 *For every* $(i, \eta, (p, q)) \in I \times \mathbb{R}_{++} \times \mathcal{M}$*, we consider the following sets:*

$B_i(p[0], q, \eta) := \{(y, z) \in Y_i \times \mathbb{R}^J : p_i[s] \cdot (y(p_i[s]) \text{-} e_i[s]) \leqslant -\eta + W(q)[s] \cdot z, \ \forall p_i \in \{p[0]\} \times P_i, \forall s \in S'_i\}$*;*

$\mathcal{A}^-(p) := \{y := (y_i) \in Y : \sum_{i=1}^{m} (y_i(p[s]) - e_i[s]) \leqslant 0, \ \forall s \in \underline{\mathbf{S}}'\}$*;*

$\mathcal{Z}^1 := \{z := (z_i) \in \mathcal{Z} : V[s_i] \cdot z_i \geqslant -1, \ \forall (i, s_i) \in I \times S_i\}$.

Assume that, for each $(i, s) \in I \times \underline{\mathbf{S}}$*, the utility index* u_i^s *is* C^1 *and such that* $\frac{\partial u_i^s}{\partial y^l}(y) > 0$*, for all* $(y, l) \in \mathbb{R}_+^{2L} \times \{1, ..., 2L\}$*. Then, there exists* $(r^1, r^2, \varepsilon_0) \in \mathbb{R}_{++}^3$*, which only depends on the fundamentals* $\mathcal{F} := ([V, (S_i)], (e_i), (u_i^s))$*, such that the following Assertions hold:*

(i) $(p \in \Lambda$ *and* $(y_i) \in \mathcal{A}^-(p)) \Rightarrow (\sum_{(i,s) \in I \times \underline{\mathbf{S}}'} \|y_i(p[s])\| < r^1)$;

(ii) $(z := (z_i) \in \mathcal{Z}^1) \Rightarrow (\|z\| := \sum_{i\in I} \|z_i\|) < r^2)$;

(iii) $((p,q) \in \mathcal{M}^*,\ p \in \cap_{i=1}^m P_i^o[\underline{\mathbf{S}}']$ *and* $\mathcal{Y}^*(p,q) \neq \varnothing) \Rightarrow (p \in \Delta_{\varepsilon_0})$;

(iv) $((i,\varepsilon,(p,q)) \in I\times\mathbb{R}_{++}\times\mathcal{M}^*$ *and* $p_i^l[s] \geqslant \varepsilon,\ \forall (p_i,l,s) \in P_i\times\{1,...,L\}\times S_i)$

$\Rightarrow (\exists \eta > 0 : B_i(p[0],q,\eta) \neq \varnothing)$.

Proof Assertion (i): Let $(p,s) \in \Lambda\times\underline{\mathbf{S}}'$, $e := \sum_{i\in I} \|e_i\|$, $r^1 := 1+me$ and $(y_i) \in \mathcal{A}^-(p)$ be given. Then, the joint relations $\sum_{i=1}^m (y_i(p[s])\text{-}e_i[s]) \leqslant 0$ and $y_i(p[s]) \geqslant 0$ hold and imply $0 \leqslant y_i(p[s]) \leqslant \sum_{j\in I} e_j[s]$, for each $i \in I$, hence, $\sum_{(i,s)\in I\times\underline{\mathbf{S}}'} \|y_i(p[s])\| < r^1$. □

Assertion (ii): Assume, by contraposition, that, for every positive integer k, there exist $z^k := (z_i^k) \in \mathcal{Z}^1$, such that $\|z^k\| := \sum_{i=1}^m \|z_i^k\| > k$. The set $\mathcal{Z}^1$ is closed, convex and contains 0 and z^k, hence, it also contains $\widetilde{z}^k := (\widetilde{z}_i^k) := \frac{z^k}{\|z^k\|}$, for every $k \in \mathbb{N}^*$, which satisfies $\|\widetilde{z}^k\| = 1$ and $V(s_i)\cdot z_i^k \geqslant -\frac{1}{k}$, for each $(i,s_i) \in I\times S_i$, by construction. The bounded sequence $(\widetilde{z}^k) \in (\mathcal{Z}^1)^{\mathbb{N}^*}$ in an Euclidean space may be assumed to converge in the closed set $\mathcal{Z}^1$ and we let $z^* := (z_i^*) := (\lim_{k\to\infty} \widetilde{z}^k) \in \mathcal{Z}^1$. From above and from the continuity of the scalar product, the following relations hold:

$\|z^*\| = 1$, $z^* \in \mathcal{Z}$ and $V[s_i]\cdot z_i^* \geqslant 0$, for each $(i,s_i) \in I\times S_i$.

Since $[V,(S_i)]$ is arbitrage-free, from Proposition 3.1 of [2], the latter relations, $z^* \in \mathcal{Z}$ and $V(S_i)z_i^* \geqslant 0$ for each $i \in I$, imply $z^* = 0$, which contradicts the former, $\|z^*\| = 1$. This contradiction proves Assertion (ii). Moreover, from the definition of $\mathcal{Z}^1$, the bound r^2 of Assertion (ii) only depends on the structure $[V,(S_i)]$). □

Assertion (iii): Let $(u_i^s)_{i\in I, s\in S_i}$ satisfy the conditions of Lemma 1 and $(p,q) \in \mathcal{M}^*$ and $[(y_i,z_i)] \in \mathcal{Y}^*(p,q) \neq \varnothing$ be given, such that $p \in \cap_{i=1}^m P_i^o[\underline{\mathbf{S}}']$. We let the reader check, as standard, that $p >> 0$. Since u_i^s is class C^1 and strictly increasing on $\mathbb{R}_+^{2L}$, for each $(i,s,(l,l')) \in I\times\underline{\mathbf{S}}\times\{1,...,2L\}^2$, the mapping $\frac{\partial u_i^s}{\partial y^l}/\frac{\partial u_i^s}{\partial y^{l'}}$ attains a maximum in $\mathbb{R}_{++}$ on the compact set $[0,r^1]^{2L}$ and we let $\beta := \sup_{i\in I, s\in\underline{\mathbf{S}}, (l,l')\in\{1,...,2L\}^2, y\in[0,r^1]^{2L}} \frac{\partial u_i^s}{\partial y^l}(y)/\frac{\partial u_i^s}{\partial y^{l'}}(y) \geqslant 1$.

Let $(s,(l,l')) \in \underline{\mathbf{S}} \times \{1,...,L\}^2$ be given, such that $l \neq l'$. Since $(y_i) \in \mathcal{A}(p)$ and $(e_i)>>0$, there exists $i \in I$, such that $y_i^l(p[s])>0$ and $(y_i[0], y_i(p[s])) \in [0, r^1]^{2L}$ along Assertion (i). We show that $\frac{p^l[s]}{p^{l'}[s]} \leqslant \beta$. Assume, by contraposition, that $\frac{p^l[s]}{p^{l'}[s]} > \beta$. Then, there exists $\varepsilon \in]0,1[$, such that $\frac{p_i^l[s]}{p_i^{l'}[s]} > \beta \geqslant \frac{\partial u_i^s(y_i(p_i[s]))}{\partial y^l} / \frac{\partial u_i^s(y_i(p_i[s]))}{\partial y^{l'}}$, $y_i^l(p_i[s]) > \varepsilon$ and $p_i^l[s] > \varepsilon$, for every $p_i[s] \in B(p[s], \varepsilon) := \{\overline{p} \in P_{(i,s)} : \|\overline{p}\text{-}p[s]\| < \varepsilon\}$. Let $y_i^* \in Y_i$ be defined by $y_i^{*\overline{l}}(p_i[\overline{s}]) := y_i^{\overline{l}}(p_i[\overline{s}])$, for every $(p_i, \overline{s}, \overline{l}) \in P_i \times S_i' \times \{1,...,L\}$, such that $\overline{s} \neq s$, or $p_i[s] \notin B(p[s], \varepsilon)$, or $\overline{l} \notin \{l, l'\}$, and $y_i^{*l}(p_i[s]) := y_i^l(p_i[s])\text{-}\varepsilon\frac{(\varepsilon - \|p_i[s]\text{-}p[s]\|)}{p_i^l[s]}$ and $y_i^{*l'}(p_i[s]) := y_i^{l'}(p_i[s]) + \varepsilon\frac{(\varepsilon - \|p_i[s]\text{-}p[s]\|)}{p_i^{l'}[s]}$ for every $p_i[s] \in B(p[s], \varepsilon)$. As is tedious, but standard, we let the reader check from the definitions and above that $(y_i^*, z_i) \in B_i(p[0], q)$ and $y_i^* \in \mathcal{R}_i(y_i)$, which contradicts the fact that $(y_i, z_i) \in B_i^*(p[0], q)$. We also let the reader check that the joint relations $\|p[s]\| = 1$ and $\frac{p^l[s]}{p^{l'}[s]} \leqslant \beta$ imply $p^l[s] \geqslant \frac{1}{\beta\sqrt{L}}$, that is, $p \in \Delta_{\varepsilon_0}$, where $\varepsilon_0 := \frac{1}{\beta\sqrt{L}}$ only depends on $\mathcal{F}$. □

Assertion (iv): Let $(i, \varepsilon, (p,q)) \in I \times \mathbb{R}_{++} \times \mathcal{M}^*$ be given, such that $p_i^l[s] \geqslant \varepsilon$, for every $(p_i, l, s) \in P_i \times \{1,...,L\} \times S_i$. From Assumptions *A1-A2*, there exists $(y, \eta) \in Y_i \times \mathbb{R}_{++}$, such that $p_i[s] \cdot (y(p_i[s])\text{-}e_i[s]) \leqslant -\eta \leqslant -\varepsilon \min_{s \in S_i'} \|e_i[s]\|$, for every $s \in S_i$. If $p[0] \neq 0$, then, from Assumption *A1*, we may choose $y[0]$ such that $-\eta < -\eta^* := p[0] \cdot (y[0]\text{-}e_i[0]) < 0$, which implies $(y, 0) \in B(p[0], q, \eta^*)$. Alternatively, if $p = 0$, then $\|q\| = 1$ and, for every $\delta \in \mathbb{R}_{++}$ small enough, the relation $(y, -\delta q) \in B_i(p[0], q, \delta)$ holds from above. □

We now state and prove Lemma 2.

Lemma 2 *For every tuple* $(i, p, s, N) \in I \times \Lambda \times S_i \times \mathbb{N}^*$, *we recall that* $G_i^n = \Pi_{s \in S_i} G_{(i,s)}^n$ *and* $P_i^o = \Lambda[0] \times P_i$, *along Definition 1, and we consider the following sets:*

$G_{(i,s)}^\infty := \cup_{n \in \mathbb{N}^*} G_{(i,s)}^n = \lim_{n \to \infty} \uparrow G_{(i,s)}^n$ *and* $G_i^\infty := \cup_{n \in \mathbb{N}^*} G_i^n = \lim_{n \to \infty} \uparrow G_i^n$;

$\mathsf{Arg}_i(p_i) := \{(p_i^n) \in (\mathbb{R}_+^{LS_i'})^{\mathbb{N}^*} : p_i = \lim_{n \to \infty} p_i^n,\ p_i^n \in \Lambda[0] \times G_i^n, \forall n \in \mathbb{N}^*\}$, *for every* $p_i \in P_i^o$.

Then, the following Assertions hold:

(i) $\forall i \in I$, $G_i^\infty \subset P_i$ *and* $\overline{G_i^\infty} = P_i$ *along the above Definition 1*;

(i') $\forall (i,n) \in I \times \mathbb{N}^*$, $(p^n, p^*) \in \Delta_{\varepsilon_0}^2 \subset P_i^o[\underline{\mathbf{S}}']^2$;

(ii) the sequence $(z^n) \in \mathcal{Z}^{\mathbb{N}}$ *is bounded, so may be assumed to converge to* $z^*{:=}(z_i^*){\in}\mathcal{Z}$;

(ii') $\exists\beta \in \mathbb{R}_{++} : \forall(i, n, s^n, l) \in I{\times}\mathbb{N}^*{\times}S_i^n{\times}\{1, ..., L\},\ x_i^{nl}[s^n] \leqslant \beta$;

(iii) $\forall(i, p_i, [(p_i^n), (p_i'^n)]) \in I{\times}P_i^o{\times}[\mathsf{Arg}_i(p_i)]^2, \exists y_i^*(p_i) \in \mathbb{R}_+^{LS_i'} : y_i^*(p^*[0]){:=}y_i^*[0]{:=}y_i^*(p_i)[0]{=}x_i^*[0]$ *and* $y_i^*(p_i[s]) := y_i^*(p_i)[s] = \lim_{n\to\infty}\ x_i^n[(s, p_i^n[s])] = \lim_{n\to\infty}\ x_i^n[(s, p_i'^n[s])]$, $\forall s \in S_i$. *Moreover, the mapping* $y_i^* : p_i \in P_i^o \mapsto y_i^*(p_i)$, *as defined above, is continuous, i.e.,* $y_i^* \in Y_i$;

(iii') $\forall(i, s) \in I{\times}\underline{\mathbf{S}}'$, $y_i^*(p^*[s]) = x_i^*[s] := \lim_{n\to\infty} x_i^n[s]$, *as defined from* (iii) *and above*;

(iv) $\forall i \in I$, $U_i(y_i^*) = \lim_{n\to\infty} U_i^n(p^n, x_i^n) \in \mathbb{R}$, *where* $y_i^* \in Y_i$ *is defined along* (iii).

Proof Assertion (i): Let $(i, s, p_i) \in I{\times}S_i{\times}P_{(i,s)}$ be given. By construction, $G_{(i,s)}^\infty \subset P_{(i,s)}$, a closed set, and there exist two sequences, $(I_{(i,s,k^n)}^n) \in (\mathcal{P}(P_{(i,s)}))^{\mathbb{N}^*}$ and $(p_i^n) \in (P_{(i,s)})^{\mathbb{N}^*}$, converging to p_i, such that $p_i^n \in G_{(i,s)}^n \cap I_{(i,s,k^n)}^n$, for each $n \in \mathbb{N}^*$, that is, $p_i \in \overline{G_{(i,s)}^\infty}$. Hence, $G_{(i,s)}^\infty \subset P_{(i,s)}$ and $P_{(i,s)} \subset \overline{G_{(i,s)}^\infty}$, for each $s \in S_i$, i.e., $G_i^\infty \subset P_i$ and $\overline{G_i^\infty} = P_i$. □

Assertion (i'): Let $n \in \mathbb{N}^*$ be given. Mutatis mutandis, all arguments of the proof of Lemma 1-(iii) apply to the n-economy, the market price $(p^n, q^n) \in \mathcal{M}^*$ and to the subset of optimal strategies in $\mathcal{Y}^n(p^n, q^n)$, which contains (x^n, z^n). Hence, $p^n \in \Delta_{\varepsilon_0}$ for the bound ε_0 of Lemma 1-(iii), which only depends on $\mathcal{F}$. From Assumption $A4$, $(p^n, p^*) \in \Delta_{\varepsilon_0}^2 \subset (P_i^o[\underline{\mathbf{S}}'])^2$, since $p^* := \lim_{n\to\infty} p^n$ and Δ_{ε_0} is closed. □

Assertion (ii): Let $n \in \mathbb{N}^*$ and $e := \sum_{i=1}^m \|e_i\|$ be given. From Assumption $A2$, the relation $(x^n, z^n) \in \mathcal{Y}^n(p^n, q^n)$ implies $\frac{z^n}{Me} \in \mathcal{Z}^1$, and, from Lemma 1-$(ii)$, $\|z^n\| \leqslant r^2 Me$. □

Assertion (ii'): Let $e := \sum_{i=1}^m \|e_i\|$, $v := \sum_{s\in S} \|V[s]\|$ and $(i, n, (p_i^n, s), l) \in I{\times}\mathbb{N}^*{\times}(G_i^n{\times}S_i){\times}\{1, ..., L\}$ be given. We let $s^n := (s, p_i^n[s]) \in \widetilde{S}_i^n$ and recall, from sub-Section 3.2, that $S_i^n := \underline{\mathbf{S}}'{\cup}\widetilde{S}_i^n$ and that $x_i^{nl}[s^n] \in \mathbb{R}_+$ (resp. $p_i^{nl}[s] \in \mathbb{R}_+$) stands for the l^{th} component of consumption $x_i^n[s^n] \in \mathbb{R}_+^L$ in state s^n (resp. of price $p_i^n[s] \in \mathbb{R}_+^L$ in state s). From Assumption $A2$ and the proof of Assertion (ii) above, the relation $(x_i^n, z_i^n) \in B_i^n(p^n[0], q^n)$ yields, in steps:

$$p_i^n[s]{\cdot}(x_i^n[s^n]\text{-}e_i[s^n]) = \sum_{l=1}^L\ p_i^l[s]{\cdot}(x_i^{nl}[s^n]\text{-}e_i^l[s]) \leqslant V^n(q^n)[s^n]{\cdot}z_i^n := V(q^n)[s]{\cdot}z_i^n \leqslant v(r^2 Me);$$

$p_i^{nl}[s]{\cdot}(x_i^{nl}[s^n]\text{-}e_i^l[s^n]) \leqslant v(r^2Me){+}\sum_{l'\in\{1,...,L\}/\{l\}} p_i^{l'}[s]{\cdot}(e_i^{l'}[s]\text{-}x_i^{nl'}[s^n]) \leqslant (vr^2)Me{+}(L\text{-}1)Me;$

$p_i^{nl}[s]{\cdot}x_i^{nl}[s^n] \leqslant (L+vr^2)Me$ and $x_i^{nl}[s^n] \leqslant \beta^o := \frac{(L+vr^2)Me}{\varepsilon}$.

Moreover, from sub-Section 3.3, for each $(i,s) \in I{\times}\underline{\mathbf{S}}'$, the sequence $(x_i^n[s]) \in (\mathbb{R}_+^L)^{\mathbb{N}^*}$ converges and is thus bounded. Let $\beta := (\beta^o + \sup_{(i,s,n)\in I\times\underline{\mathbf{S}}'\times\mathbb{N}^*} x_i^{nl}[s]) \in \mathbb{R}$. Then, from above, the relation $x_i^{nl}[s^n] \leqslant \beta$ holds for every tuple $(i,n,s^n,l) \in I{\times}\mathbb{N}^*{\times}S_i^n{\times}\{1,...,L\}$. $\square$

Assertion (iii)-(iii'): Let $(i,p_i,s,(p_i^n),(p_i'^n)) \in I{\times}P_i^o{\times}S_i{\times}\mathsf{Arg}_i(p_i){\times}\mathsf{Arg}_i(p_i)$ be given. From Assertion (i), $\mathsf{Arg}_i(p_i)$ is non-empty. From sub-Section 3.3, the sequence $(x_i^n[0]) \in (\mathbb{R}_+^L)^{\mathbb{N}^*}$ converges towards $x_i^*[0] \in \mathbb{R}_+^L$. We let $y_i^*[0] := x_i^*[0]$, and, for every $(x^o,p,z) \in \Theta := \mathbb{R}_+^L{\times}\mathbb{R}_+^L{\times}\mathbb{R}^J$:

$B_i^s(p,z) := \{x[s] \in \mathbb{R}_+^L : p{\cdot}(x[s]\text{-}e_i[s]) \leqslant V[s]{\cdot}z\};$

$\varphi_i^s(x^o,p,z) = \arg\max_{x[s]\in B_i^s(p,z)} u_i^s(x^o,x[s]).$

From the conditions of strict concavity and continuity of u_i^s of Assumption *A3*, it is standard that the correspondence $\varphi_i^s : (x^o,p,z) \mapsto \varphi_i^s(x^o,p,z)$ is, first, a mapping from Θ to $\mathbb{R}_+^L$ and, second, continuous on Θ (see, e.g., [8], p. 19).

By construction, for every $n \in \mathbb{N}^*$, the relations $p_i^n[s] \in P_{(i,s)}$, $\pi_{(i,s)}^n(p_i^n[s]) > 0$ (respectively, $p_i'^n[s] \in P_{(i,s)}$, $\pi_{(i,s)}^n(p_i'^n[s]) > 0$) hold and, moreover, $x_i^n[(s,p_i^n[s])] = \varphi_i^s(x_i^n[0],p_i^n[s],z_i^n)$ (resp. $x_i^n[(s,p_i'^n[s])] = \varphi_i^s(x_i^n[0],p_i'^n[s],z_i^n)$), since $\mathcal{C}^n$ is an equilibrium in the n-economy. From sub-Section 3.3 and above the sequence of elements $(x_i^n[0],p_i^n[s],p_i'^n[s],z_i^n)$ (for $n \in \mathbb{N}^*$) converges to $(x_i^*[0],p_i[s],p_i[s],z_i^*)$. The fact that φ_i^s is a continuous mapping insures the existence of a unique vector $y_i^*(p_i[s]) := \varphi_i^s(x_i^*[0],p_i[s],z_i^*)$, which satisfies, from above: $y_i^*(p_i[s]) = \lim_{n\to\infty} x_i^n[(s,p_i^n[s])] = \lim_{n\to\infty} x_i^n[(s,p_i'^n[s])]$.

The relations $p_i[0] \mapsto y_i^*(p_i[0]) := y_i^*[0]$, and $p_i[s] \mapsto y_i^*(p_i[s])$, defined from above for every $(s,p_i) \in S_i{\times}P_i^o$, yield a mapping $y_i^* : p \in P_i^o \mapsto y_i^*(p) := (y_i^*(p[s]))_{s\in S_i'} \in \mathbb{R}_+^{LS_i'}$, which is constant in the first period (i.e., state $s=0$) component and continuous (from the continuity of φ_i^s for each $s \in S_i$), that is, $y_i^* \in Y_i$. This proves Assertion (iii).

Let $(i,s) \in I \times \underline{\mathbf{S}}$ be given. The fact that $\mathcal{C}^n$ is an n-equilibrium yields, for each $n \in \mathbb{N}^*$, $x_i^n[s] = \varphi_i^s(x_i^n[0], p^n[s], z_i^n)$, hence, in the limit, $x_i^*[s] = \varphi_i^s(x_i^*[0], p^*[s], z_i^*) := y_i^*(p^*[s])$, from above and the continuity of the mapping φ_i^s. This proves Assertion (iii'). $\square$

Assertion (iv): Let $(i,n) \in I \times \mathbb{N}^*$ be given and define $\widetilde{y}_i^n \in \mathbb{R}_+^L \times (\mathbb{R}_+^{LS_i})^{P_i}$ as follows:

$\widetilde{y}_i^n[0] := x_i^n[0]$;

$\widetilde{y}_i^n(p) := x_i^n[s]$ for every $(s,p) \in \underline{\mathbf{S}} \times B_i^{r^n}(p^n[s])$ and, along sub-Section 3.1,

$\widetilde{y}_i^n(p) := x_i^n[(s, g_{(i,s,k^n)}^n)]$ for every $(s,k^n,p) \in \underline{\mathbf{S}} \times K_{(i,s)}^n \times I_{(i,s,k^n)}^n / B_i^{r^n}(p^n[s])$;

$\widetilde{y}_i^n(p) := x_i^n[(s, g_{(i,s,k^n)}^n)]$ for every $(s,k^n,p) \in S_i / \underline{\mathbf{S}} \times K_{(i,s)}^n \times I_{(i,s,k^n)}^n$.

By construction, one has: $U_i(\widetilde{y}_i^n) := \sum_{s \in S_i} \int_{p \in P_i} u_i^s(\widetilde{y}_i^n[0], \widetilde{y}_i^n(p[s])) d\pi_{(i,s)}(p[s])$

$:= \sum_{s \in \underline{\mathbf{S}}} \pi_{(i,s)}(B_i^{r^n}(p^n[s])) u_i^s(x_i^n[0], x_i^n[s]) + \sum_{s \in S_i / \underline{\mathbf{S}}} \sum_{k^n \in K_{(i,s)}^n} \pi_{(i,s)}(I_{(i,s,k^n)}^n) u_i^s(x_i^n[0], x_i^n[(s, g_{(i,s,k^n)}^n)])$

$+ \sum_{s \in \underline{\mathbf{S}}} \sum_{k^n \in K_{(i,s)}^n} \pi_{(i,s)}(I_{(i,s,k^n)}^n / B_i^{r^n}(p^n[s])) u_i^s(x_i^n[0], x_i^n[(s, g_{(i,s,k^n)}^n)]) := U_i^n(p^n, x_i^n)$.

From Assumption *A3*, Lemma 1-(i) and Lemma 2-(ii'), the mapping $p \in P_{(i,s)} \mapsto \varphi_{(i,s)}^n(p) := u_i^s(\widetilde{y}_i^n[0], \widetilde{y}_i^n(p)) \in \mathbb{R}_+$ is uniformly bounded for $(s,n) \in S_i \times \mathbb{N}^*$. From Lemma 2-$(iii)$ and the continuity of u_i^s, for every $(s,p) \in S_i \times P_{(i,s)}$, the sequence $(\varphi_{(i,s)}^n(p)) \in \mathbb{R}^{\mathbb{N}^*}$ converges to $\varphi_{(i,s)}(p) := u_i^s(y_i^*[0], y_i^*(p))$ by construction. Hence, for every $s \in S_i$, the sequence of mappings $(\varphi_{(i,s)}^n)$ and mapping $\varphi_{(i,s)} : p \in P_{(i,s)} \mapsto u_i^s(y_i^*[0], y_i^*(p))$ are bounded and measurable with respect to $\pi_{(i,s)}$ and meet the conditions of Lebesgue's theorem of dominated convergence (see, e.g., [12], p. 59], which yields, from above:

$\int_{p \in P_i} \varphi_{(i,s)}(p[s]) d\pi_{(i,s)}(p[s]) = [\lim_{n \to \infty} \int_{p \in P_i} \varphi_{(i,s)}^n(p[s]) d\pi_{(i,s)}(p[s])] \in \mathbb{R}$;

$\sum_{s \in S_i} \int_{p \in P_i} \varphi_{(i,s)}(p[s]) d\pi_{(i,s)}(p[s]) = [\lim_{n \to \infty} \sum_{s \in S_i} \int_{p \in P_i} \varphi_{(i,s)}^n(p[s]) d\pi_{(i,s)}(p[s])] \in \mathbb{R}$;

that is, $U_i(y_i^*) := \lim_{n \to \infty} U_i^n(p^n, y_i^n) \in \mathbb{R}$. $\square$

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